UNIVERSITY OF PENNSYLVANIA

ELLIPTIC ORTHOGONAL POLYNOMIALS

A DISSERTATION

IN MATHEMATICS

PRESENTED TO THE FACULTY OF THE GRADUATE SCHOOL IN
PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

CARL JOHN REES

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By C. J. Rees

Introduction. Elliptic orthogonal polynomials (OP) may be defined as the orthogonal polynomials $\varphi_n(x)$, of degree $n = 0, 1, \cdots$, which satisfy the relations

$$\int_{-1}^{1} \varphi_n(x)\varphi_m(x) \frac{dx}{X^{\frac{1}{2}}} = \delta_{m,n} \qquad (m, n = 0, 1, \cdots),$$

(1)

$$X(x, k) = (1 - x^2)(1 - k^2x^2) \qquad (0 < k < 1).$$

The limiting case $k = 0$ yields trigonometric polynomials.

The above definition of elliptic polynomials is different from that adopted by Achyeser [1], [2], [3].

In Part I of this paper we study the associated moments for which a recurrence relation is derived and an interesting orthogonality property appears. In Part II we derive for $\varphi_n(x)$ a linear homogeneous differential equation of the second order.

Heine [8; 294] derived such a differential equation for elliptic orthogonal polynomials with the weight function $(x(x - \alpha)(x - \beta))^{-\frac{1}{2}}$. The coefficients of Heine's equation however depend upon three parameters, two of which are given in terms of a third as roots of two algebraic equations, each of degree $2n + 1$. Thus Heine's differential equation is rather an existence proof and can hardly be used for a further study of elliptic polynomials.

Employing the method of Shohat [15], we find explicitly the differential equation for elliptic polynomials, with weight function $X^{-\frac{1}{2}}$ whose coefficients depend upon one parameter only; namely, λ_n—which plays such an important rôle in the theory of orthogonal polynomials—and for which we give a recurrence relation.

I. Elliptic Moments

1. The function

$$\int_{-1}^{x} \frac{dx}{((1 - x)^2(1 - k^2x^2))^{\frac{1}{2}}} \equiv \int_{-1}^{x} \frac{dx}{(X(x))^{\frac{1}{2}}} \qquad (0 < k < 1)$$

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is the unique solution in $(-1, 1)$ of the moment problem:

$$\int_{-1}^{1} x^i \, d\psi(x) = \alpha_i = \int_{-1}^{1} \frac{x^i \, dx}{(X(x))^{\frac{1}{2}}} \qquad (i = 0, 1, 2, \cdots),$$

$\psi(x)$ bounded, non-decreasing in $(-1, 1)$, $\psi(-1) = 0$. We call these α_i *elliptic moments*. Here we have the "symmetric" case, that is,

$$\alpha_{2n+1} = 0, \qquad \alpha_{2n} = 2\int_0^1 \frac{x^{2n}\,dx}{X^{\frac{1}{2}}} = 2\int_0^{\frac{1}{2}\pi} \frac{\sin^{2n}\theta\,d\theta}{(1 - k^2\sin^2\theta)^{\frac{1}{2}}} \equiv \beta_n \tag{2}$$

$$(n = 0, 1, \cdots).$$

By means of Tchebicheff and Schwartz inequalities we find at once

$$\begin{aligned} &\beta_n\beta_{n+m} > \beta_{n+m-j}\beta_{j+m} && (n = 2, 3, \cdots; j = 1, 2, \cdots, n-1), \\ &\beta_{n+m}^2 \leq \beta_{2n+m-h}\beta_{h+m} && (h = 1, 2, \cdots m; m = 0, 1, \cdots, n). \end{aligned} \tag{3}$$

It will be shown later that any β_i $(i \geq 2)$ is a linear combination of β_0 and β_1, where

$$\beta_0 = \int_0^{\frac{1}{2}\pi} \frac{d\theta}{(1 - k^2\sin^2\theta)^{\frac{1}{2}}} \equiv 2K, \tag{4}$$

$$\beta_1 = 2\int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta\,d\theta}{(1 - k^2\sin^2\theta)^{\frac{1}{2}}} \equiv 2\frac{K - E_1}{k^2} \tag{5}$$

$$\beta_0 = \pi, \qquad 2\beta_1 = \pi, \qquad E_1 = K = \tfrac{1}{2}\pi \text{ for } k = 0.$$

K and E_1 are the complete elliptic integrals of the first and second kinds.

Expanding (4) and (5) in powers of k^2 yields

$$K - \frac{\pi k^2}{16} \geq \beta_0 - \beta_1 \geq E_1 + \frac{3\pi k^2}{16},$$

the equalities holding only when $k = 0$.

Evidently

$$2K = \beta_0 > \beta_1 \cdots \beta_n > \cdots, \qquad \lim_{n\to\infty} \beta_n = 0. \tag{6}$$

Furthermore (3), with $m = 0$, $h = n - 1$, yields, by (6),

$$\beta_n^2 \leq \beta_{n+1}\beta_{n-1}, \qquad 1 > \frac{\beta_1}{\beta_0} \geq \frac{\beta_2}{\beta_1} \geq \cdots, \tag{7}$$

$$\lim_{n\to\infty} \frac{\beta_n}{\beta_{n-1}} = l \leq 1.$$

It will be shown later that $l = 1$.

From the theory of elliptic functions [4; 277ff] we know that

$$(2n+1)k^2\int_0^K \mathrm{sn}^{2n+2}u\,du - 2n(1+k^2)\int_0^K \mathrm{sn}^{2n}u\,du + (2n-1)\int_0^K \mathrm{sn}^{2n-2}u\,du$$

$$= \int_0^K \mathrm{sn}^{2n-3}u\,\mathrm{cn}\,u\,\mathrm{dn}\,u\,du = 0 \qquad (n = 0, 1, 2, \cdots).$$

Whence, letting $x = \mathrm{sn}\,u$, we get the following recurrence relation, of importance in the subsequent discussion:

$$(8) \qquad (2n+3)k^2\beta_{n+2} - (2n+2)(1+k^2)\beta_{n+1} + (2n+1)\beta_n = 0.$$

It yields at once a continued fraction for the ratio β_n/β_{n-1} :

$$\frac{\beta_n}{\beta_{n-1}} = \frac{(2n-2)(1+k^2)}{(2n-1)k^2} - \cfrac{1}{\cfrac{(2n-1)(2n-4)(1+k^2)}{(2n-3)^2}}$$

$$- \cfrac{1}{\cfrac{(2n-3)^2(2n-6)(1+k^2)}{(2n-1)(2n-5)^2k^2}} - \cfrac{1}{\cfrac{(2n-1)(2n-5)^2(2n-8)(1+k^2)}{(2n-3)^2(2n-7)^2}}$$

$$- \cdots \cfrac{1}{-r_n(1+k^2)\dfrac{\beta_1}{\beta_0}},$$

$$r_n = \frac{(2n-1)(2n-5)^2(2n-9)^2\cdots 7^2\cdot 3^2}{(2n-3)^2(2n-7)^2\cdots 9^2\cdot 5^2} \qquad (n \text{ even}),$$

$$r_n = \frac{(2n-3)^2(2n-7)^2\cdots 7^2\cdot 3^2}{(2n-1)(2n-5)^2(2n-9)^2\cdots 9^2\cdot 5^2k^2} \qquad (n \text{ odd}).$$

2. We proceed to derive a more general recurrence relation for the β_n giving (8) as a special case. Denoting by ν an arbitrary constant, write

$$\frac{d}{dx}\left(\frac{x^{2n+1}X^{\frac{1}{2}}}{(1+\nu x^2)^{m-1}}\right) = \frac{-2(1-m)[x^{2n} - (1+k^2)x^{2n+2} + k^2x^{2n+4}]}{(1+\nu x^2)^m X^{\frac{1}{2}}}$$

$$+ \frac{[2(2n+3) + 2(1-m)]k^2x^{2n+4} + 2(n-m+\frac{3}{2})x^{2n}}{(1+\nu x^2)^{m-1}X^{\frac{1}{2}}}$$

$$- \frac{2[n+1+(1-m)](1+k^2)x^{2n+2}}{(1+\nu x^2)^{m-1}X^{\frac{1}{2}}}.$$

Whence

$$[2(n-m)+3]\int_{-1}^{1}\frac{x^{2n}\,dx}{(1+\nu x^2)^{m-1}X^{\frac{1}{2}}} - 2[(n-m)+2](1+k^2)\int_{-1}^{1}\frac{x^{2n+2}\,dx}{(1+\nu x^2)^{m-1}X^{\frac{1}{2}}} + [2(n-m)+5]k^2\int_{-1}^{1}\frac{x^{2n+4}\,dx}{(1+\nu x^2)^{m-1}X^{\frac{1}{2}}} + 2(m-1)\int_{-1}^{1}\frac{x^{2m}\,dx}{(1+\nu x^2)^{m}X^{\frac{1}{2}}} - 2(m-1)(1+k^2)\int_{-1}^{1}\frac{x^{2n+2}\,dx}{(1+\nu x^2)^{m}X^{\frac{1}{2}}} + 2(m-1)k^2\int_{-1}^{1}\frac{x^{2n+4}\,dx}{(1+\nu x^2)^{m}X^{\frac{1}{2}}} = 0.$$

This relation holds for all positive n and m and for any ν complex or real ($|\nu|<1$).

Taking $m=1$ we see that β_n is a solution of the hypergeometric difference equation [5; 68–127].

$$(9)\qquad (2n+3)k^2y_{n+2} - 2(n+1)(1+k^2)y_{n+1} + (2n+1)y_n = 0.$$

A second solution of (9) can be readily found either through its characteristic equation or by treating it as a Poincaré difference equation [9; 301 ff].

3. **Elliptic moments of other kinds.** If in

$$\int_{-1}^{1} x^n\,d\psi(x) = \alpha_n \qquad (n=0,1,2,\cdots)$$

in addition to the distribution $d\psi(x) = X^{-\frac{1}{2}}\,dx$ we use

$$d\psi_1(x) = \left(\frac{1-k^2x^2}{1-x^2}\right)^{\frac{1}{2}}dx, \qquad d\psi_2(x) = \frac{dx}{(1+\nu x^2)X^{\frac{1}{2}}},$$

where ν is real, $|\nu|<1$, we obtain two new sets of moments closely related to $\{\beta_n\}$:

$$\gamma_n = 2\int_0^1\left(\frac{1-k^2x^2}{1-x^2}\right)^{\frac{1}{2}}x^{2n}\,dx, \qquad \delta_n = 2\int_0^1\frac{x^{2n}\,dx}{(1+\nu x^2)X^{\frac{1}{2}}}$$

with

$$(10)\qquad \gamma_n = \beta_n - k^2\beta_{n+1}, \quad \beta_n = \delta_n + \nu\delta_{n+1} \qquad (n=0,1,\cdots).$$

They furnish a second simple method of obtaining the recurrence relation (8) for β_n. It suffices to integrate the expression for γ_n by parts and, using (10), to eliminate γ_n, γ_{n-1}. Similarly we may obtain recurrence relations for γ_n and δ_n.

4. **Difference-differential equation for β_n.** Use the difference equation [16; 189]

$$(11) \qquad (2n+1)(k^4-k^2)J_{n+1} + 2n(2k^2-1)J_n + (2n-1)J_{n-1} = 0$$

satisfied by

$$J_n = \int_0^{\frac{1}{2}\pi} \frac{\sin^{2n}\theta\, d\theta}{(1-k^2\sin^2\theta)^{\frac{1}{2}(2n+1)}},$$

and observe that

$$\frac{d^n\beta_0}{d(k^2)^n} \equiv \beta_0^{[n]} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^{n-1}} J_n .$$

Then (11) gives

$$4k^2k'^2\beta_0^{[n+1]} + 4n(2k^2-1)\beta_0^{[n]} + (2n-1)^2\beta_0^{[n-1]}.$$

Taking $\nu = -k^2$ in the δ's of (10) and making use of the relation

$$\frac{\partial\beta_n}{\partial k} = 2k\int_0^1 \frac{x^{2n+2}\,dx}{(1-k^2x^2)X^{\frac{1}{2}}},$$

we obtain the following mixed difference-differential equation for β_n:

$$\beta_n = \frac{1}{k}\frac{\partial\beta_{n-1}}{\partial k} - k\frac{\partial\beta_n}{\partial k}.$$

5. **Asymptotic properties of β_n $(n\to\infty)$.** By (2),

$$\beta_n \geq 2\int_0^{\frac{1}{2}\pi} \sin^{2n}\varphi\, d\varphi = 2\pi^{\frac{1}{2}}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}.$$

For n very large we have more precisely [12], [6; 8]

$$\beta_n = \frac{\Gamma(\frac{1}{2})+O(1)}{2^{\frac{1}{2}}k'n^{\frac{1}{2}}} \sim (\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{1}{k'n^{\frac{1}{2}}},$$

$$\beta_n \sim \frac{\Gamma(\frac{1}{2})}{2^{\frac{1}{2}}k'n^{\frac{1}{2}}}\left[1 - \frac{k'^2-k^2}{2k'^2n} + \frac{3(19k^4-14k^2+4)}{8k'^4n^2} + \cdots\right].$$

Whence

$$(12)\qquad \frac{\beta_n}{\beta_{n-1}} \sim 1 - \frac{\frac{1}{2}}{n} - \frac{3/2^4}{n^2} - \frac{21k'^4 - 16k'^2k^2 + 12(19k^4 - 14k^2 + 4)}{n^3} + \cdots.$$

We may also obtain an asymptotic expression for β_n from the general theory of the hypergeometric difference equation.

6. **Expression for β_n in terms of certain fundamental constants.** Elliptic moments may be written as a linear combination of the two fundamental moments β_0 and β_1 by solving (8) as a system of linear equations and using the properties of continuants. It turns out to be more advantageous to express β_n in terms of the two constants K and E_k where

$$(13)\qquad E_k \equiv \frac{\partial E_1}{\partial k} = \frac{1}{k}(E_1 - K).$$

Rewrite the recurrence relation (8) as

$$(14)\qquad \beta_{n+2} = \frac{1}{k}\left\{2\frac{2n+2}{2n+3}r\beta_{n+1} - \frac{2n+1}{2n+3}\frac{1}{k}\beta_n\right\},$$

where

$$r = \tfrac{1}{2}\left(k + \frac{1}{k}\right) \qquad (k > 0).$$

It follows, since $\beta_0 = 2K$, $\beta_1 = 2k^{-2}(E_1 - K) = -2k^{-1}E_k$, that

$$(15)\qquad \beta_n = -\frac{2^{n-1}}{k^n}B(n, \tfrac{1}{2})\{Q_{n-1}(r)E_k + P_{n-1}(r)K\} \quad (n = 2, 3, \cdots),$$

where Q_i *and* P_i $(i = 1, 2, \cdots)$ *are polynomials in* r *of degree* i *and* $i - 1$ *respectively.*

Substituting (15) into (14) we must have identically

$$(16)\qquad \left\{Q_{n+1} - rQ_n + \frac{(2n+1)^2}{2^4n(n+1)}Q_{n-1}\right\}E_k + \left\{P_{n+1} - rP_n + \frac{(2n+1)^2}{2^4n(n+1)}P_{n-1}\right\}K = 0,$$

for $n \geq 2$. But, from (15), E_kK^{-1} cannot be a rational function of r, hence of k. Thus (16) yields the following result:

The polynomials $Q_n \equiv Q_n(r)$ *and* $P_n \equiv P_n(r)$ *both satisfy the following recurrence relation:*

$$(17)\qquad y_n = ry_{n-1} - \frac{(2n-1)^2}{2^4n(n-1)}y_{n-2} \qquad (n \geq 2),$$

that is,

$$Q_n \equiv rQ_{n-1} - \frac{(2n-1)^2}{2^4 n(n-1)} Q_{n-2},$$

$$P_n \equiv rP_{n-1} - \frac{(2n-1)^2}{2^4 n(n-1)} P_{n-2} \qquad (n \geq 2), \tag{18}$$

$$Q_0 = 1,\ Q_1 = r; \qquad P_1 = \tfrac{1}{4},\ P_2 = \tfrac{1}{4}r.$$

This leads to further conclusions. First, (18) shows that $Q_n(r)$ and $P_n(r)$ are respectively the denominator and numerator of the n-th approximant to the continued fraction

$$F(r) \equiv \cfrac{\frac{1}{4}}{r - \cfrac{\frac{3^2}{2^4 \cdot 1 \cdot 2}}{r - \cfrac{\frac{5^2}{2^4 \cdot 2 \cdot 3}}{r - \cdots - \cfrac{\frac{(2n-1)^2}{2^4 n(n-1)}}{r - \cdots}}}} \tag{19}$$

Moreover, (17) is a recurrence relation of the form

$$y_n + (c_m - r)y_{n-1} + \lambda_n y_{n-1} \qquad (n \geq 2) \tag{20}$$

with

$$\lambda_n = \frac{(2n-1)^2}{2^4 n(n-1)} > 0 \qquad (c_n = 0),$$

and this is a necessary and sufficient condition for the orthogonality of the polynomials involved [7], [14]. We thus arrived at

THEOREM I. *Both sequences* $\{Q_n(r)\}$ $(n = 0, 1, \cdots)$ *and* $\{P_n(r)\}$ $(n = 1, 2, \cdots)$, *as polynomials in* r, *are orthogonal sequences, that is, there exist two non-decreasing bounded functions* $\alpha(r)$ *and* $\beta(r)$ *such that*

$$\int_{-\infty}^{\infty} Q_n(r)Q_m(r)\, d\alpha(r) = 0 \qquad (m, n = 0, 1, \cdots ; m \neq n),$$

$$\int_{-\infty}^{\infty} P_n(r)P_m(r)\, d\beta(r) = 0. \tag{21}$$

Furthermore, the "true interval of orthogonality" [14] *is the same and finite for each sequence and is symmetric to* 0.

The latter statement results from the following facts: (i) $\{\lambda_n\}$ is a bounded sequence and we are dealing here with the "symmetric" case (all $c_n = 0$), see [12]; (ii) $\{P_n\}$ and $\{Q_n\}$ are the numerators and denominators of the same continued fraction (19), see [10], [11].

The normalization is very simple. The normalizing factors for $\{Q_n\}$ and $\{P_n\}$ respectively are

$$(22)\qquad \begin{aligned} a_n &= \frac{1}{(\lambda_1\lambda_2\cdots\lambda_n)^{\frac{1}{2}}} = \frac{2^{3n+1}(n+1)^{\frac{1}{2}}n!^2}{(2n+1)!} = 2^n\pi^{\frac{1}{2}}\left[1+O\left(\frac{1}{n}\right)\right], \\ a_n' &= \frac{2^{3n+1}(n+2)^{\frac{1}{2}}(n+1)n!^2}{(2n+3)!} = 2^{n-1}\pi^{\frac{1}{2}}\left[1+O\left(\frac{1}{n}\right)\right]. \end{aligned}$$

THEOREM II. *The true interval of orthogonality in Theorem* I *is* $(-1, 1)$.

First proof. We have here

$$\lim_{n\to\infty}\lambda_n = \tfrac{1}{4}.$$

By a known theorem concerning orthogonal polynomials [13] the zeros of $Q_n(r)$ are everywhere dense in $(-1, 1)$. Hence, the true interval of orthogonality (which contains all above zeros) is, say, $(-h, h)$, and $h \geq 1$.

Assume now that $h > 1$. For n sufficiently large there is a zero of Q_{n-1}, say r_0, close to $(-h)$ so that $r_0 < -1$. Since the zeros of P_{n-1} separate those of Q_{n-1}, the smallest zero of P_{n-1} is on the right of r_0. But $r_0 < -1$; hence, for the corresponding value of k, say

$$k_0\left(\frac{1}{r_0} = \frac{1}{2}\left(k_0 + \frac{1}{k_0}\right)\right),$$

$P_{n-1}(r_0)$ has the same sign as $P_{n-1}(-\infty)$, namely, $(-1)^{n-2}$. Then, by (15), the highest coefficient of $P_{n-1}(r)$ being negative,

$$\operatorname{sgn}\beta_n(k_0) = (-1)^{2n-1} < 0, \qquad \beta_n(k_0) \equiv \beta_n(-k_0) < 0,$$

which is impossible since $\beta_n > 0$ for $0 < -k_0 < 1$.

Second proof. In the continued fraction (19), the limit, as $n \to \infty$, of the partial numerator is $-\frac{1}{4}$. By a theorem of Perron [10; 288], $F(r)$ converges in the whole complex r-plane save the cut $-1 \leq r \leq 1$. On the other hand $F(r)$ is "associated" with the integral

$$\int_{-h}^{h}\frac{d\alpha(y)}{r-y}.$$

It now follows from Markoff's theorem [10; 385] that $h = 1$ (since we already know that $h \geq 1$).

It is seen that (21) can be written as

$$\int_{-1}^{1} Q_n(r)Q_m(r)\,d\alpha(r) = 0 \qquad (m, n = 0, 1, \cdots; m \neq n),$$

$$\int_{-1}^{1} P_n(r)P_m(r)\,d\beta(r) = 0 \qquad (m, n = 1, 2, \cdots; m \neq n).$$

We proceed to find the value of the continued fraction in (19) in explicit form in terms of elliptic integrals. From (12) and (15),

$$\frac{\beta_{n+1}}{\beta_n} = 1 + O\left(\frac{1}{n}\right) = \frac{2^2 n}{(2n+1)k}\frac{Q_n(r)}{Q_{n-1}(r)}\left[\frac{E_k + \dfrac{P_n(r)}{Q_n(r)}K}{E_k + \dfrac{P_{n-1}(r)}{Q_{n-1}(r)}K}\right] \tag{23}$$

From the theory of OP, see [13], for every r real or complex, not on $(-1, 1)$

$$\lim_{n\to\infty}\frac{Q_n(r)}{Q_{n-1}(r)} = \frac{1}{2k'} \tag{24}$$

where $1/2k$ is the zero of largest modulus of the characteristic equation for (17). Since $0 < k < 1$, we agree to take, in the relation, $2r = k + 1/k$,

$$k = r - (r^2 - 1)^{\frac{1}{2}}, \qquad \frac{1}{k} = r + (r^2 - 1)^{\frac{1}{2}}.$$

Letting $n \to \infty$ in (23) find, by (24), for r real or complex not on $(-1, 1)$

$$\lim_{n\to\infty}\left[\frac{E_k + \dfrac{P_n(r)}{Q_n(r)}K}{E_k + \dfrac{P_{n-1}(r)}{Q_{n-1}(r)}K}\right] = k^2. \tag{25}$$

On the other hand, by Markoff's theorem (used above)

$$\lim_{n\to\infty}\frac{P_n(r)}{Q_n(r)} = F(r),$$

see (19). *This limit is precisely* $-E_kK^{-1}$. Assume the contrary. Then for $r > 1$, (25) gives

$$k^2 = \frac{E_k + F(r)K}{E_k + F(r)K} = 1,$$

which is clearly impossible. Thus for all r, real or complex, not on $(-1, 1)$

$$\int_{-1}^{1}\frac{d\alpha(y)}{r-y} = \frac{1}{r} - \cfrac{\dfrac{3^2}{2^4\cdot 1\cdot 2}}{r} - \cfrac{\dfrac{5^2}{2^4\cdot 2\cdot 3}}{r} - \cdots = -\frac{E_k}{K} \qquad \left(2r = k + \frac{1}{k},\ |\,k\,| < 1\right).$$

To determine $\alpha(y)$ at all its points of continuity one would employ the inversion formula of Perron [10].

7. Relation of Q_n to Legendre polynomials. Write

$$\beta_n = \frac{2}{k'}\{Q_n'(r)E_k + P_{n-1}'(r)K\} \qquad (n = 1, 2, \cdots),$$

$$Q_n'(r) \equiv B(n+1, \tfrac{1}{2})Q_n(r), \qquad P_n'(r) \equiv B(n+1, \tfrac{1}{2})P_n(r)$$

and find as above

$$Q'_{n+1} = 2\frac{2n+2}{2n+3} rQ'_n - \frac{2n+1}{2n+3} Q'_{n-1} \qquad (n = 1, 2, \cdots)$$

(26)

$$Q'_0 = 1, \qquad Q'_1 = \frac{4r}{3}.$$

The difference relation (26) may be written

$$(n + \tfrac{3}{2})Q'_{n+1} - 2(n+1)rQ'_n + (n + \tfrac{1}{2})Q'_{n-1} = 0. \tag{27}$$

If we substitute

$$n + \tfrac{1}{2} = n', \qquad Q'_n = Q'_{n'-\frac{1}{2}} \equiv y_n$$

and drop the primes, (27) becomes

$$(n+1)y_{n+1} - (2n+1)ry_n + ny_{n-1} = 0,$$

which is precisely the recurrence relation satisfied by Legendre polynomials. A similar consideration applies to P'_n.

Moreover, in the recurrence relation (20) written for Legendre polynomials

$$c_n = 0, \qquad \lambda_n = \frac{(n-1)^2}{(2n-3)(2n-1)}.$$

If we here replace n by $n + \frac{1}{2}$ and drop the primes, λ_n becomes

$$\frac{(n+\frac{1}{2})^2}{2n(2n-2)} = \frac{(2n-1)^2}{2^4 n(n-1)},$$

which is the λ_n for Q_n. It follows that the same holds for the normalizing factors.

II. Differential Equation for Elliptic Polynomials

1. **Relations for elliptic** OP. Write the elliptic OP with the highest coefficient unity in the form

$$\phi_n(x) = x^n + d_{n,n-2}\, x^{n-2} + d_{n,n-4}\, x^{n-4} + \cdots . \tag{28}$$

By means of the orthogonality relation and of integration by parts we find

$$\int_{-1}^{1} \phi'_n(x) G_{n-4}(x) X^{\frac{1}{2}}\, dx = 0, \tag{29}$$

where $G_s(x)$ stands for an arbitrary polynomial of degree $\leq s$. The same orthogonality relation yields

$$\int_{-1}^{1} \phi_n(x)x^n \frac{dx}{X^{\frac{1}{2}}} = \frac{1}{a_n^2},$$

$$\int_{-1}^{1} \phi_n(x)x^{n+2} \frac{dx}{X^{\frac{1}{2}}} = -\frac{d_{n+2,n}}{a_n^2}, \tag{30}$$

$$\int_{-1}^{1} \phi_n(x)x^{n+4} \frac{dx}{X^{\frac{1}{2}}} = -\frac{1}{a_n^2}\{d_{n+4,n+2}d_{n+2,n} - d_{n+4,n}\};$$

$$\int_{-1}^{1} \phi_n'(x)x^{n-3}X^{\frac{1}{2}}\,dx = -\frac{(n-1)k^2}{a_n^2},$$

$$\int_{-1}^{1} \phi_n'(x)x^{n-1}X^{\frac{1}{2}}\,dx = \frac{1}{a_n^2}\{(n+1)k^2d_{n+2,n} + n(1+k^2)\}, \tag{31}$$

$$\int_{-1}^{1} \phi_n'(x)x^{n+1}X^{\frac{1}{2}}\,dx = \frac{1}{a_n^2}\{(n+3)k^2[d_{n+4,n} - d_{n+4,n+2}d_{n+2,n}] - (n+2)(1+k^2)d_{n+2,n} - (n+1)\};$$

$$\int_{-1}^{1} \phi_n'(x)\phi_{n-1}(x)X^{\frac{1}{2}}\,dx = \frac{1}{a_n^2}\{(n-1)k^2[d_{n+2,n} - d_{n-1,n-3}] + 2k^2d_{n+2,n} + n(1+k^2)\},$$

$$\int_{-1}^{1} \phi_n'(x)\phi_{n+1}(x)X^{\frac{1}{2}}\,dx = \frac{1}{a_n^2}\{(n+3)k^2[d_{n+4,n} - d_{n+4,n+2}d_{n+2,n}] - (n+2)(1+k^2)d_{n+2,n} - (n+1) - (n+1)k^2d_{n+1,n-1}d_{n+2,n} + n(1+k^2)d_{n+1,n-1} - (n-1)k^2d_{n+1,n-3}\}; \tag{32}$$

$$\int_{-1}^{1} \phi_{n-1}(x)\phi_n'X^{\frac{1}{2}}\,dx = \frac{1}{a_n^2}\{n - (n-2)(1+k^2)d_{n,n-2} + (n-4)k^2d_{n,n-4} + k^2d_{n+1,n-1}[nd_{n+3,n-1} - (n-2)d_{n,n-2}] + n(1+k^2)d_{n+1,n-1} - nk^2d_{n+3,n-1}\}. \tag{33}$$

Substitute (28) in the recurrence relation (20), with $c_n = 0$, and equate the coefficients of x^{n-1}, x^{n-4} respectively. We get [15]

$$d_{n-1,n-3} - d_{n,n-2} = \lambda_n\,, \tag{34}$$

$$d_{n-1,n-5} - d_{n,n-4} = \lambda_n d_{n-2,n-4} = \lambda_n(\lambda_2 + \lambda_3 + \cdots + \lambda_{n-2}). \tag{35}$$

Relations (29), (31), (32), (33) hold for elliptic polynomials only, while (34), (35), (30) are true for all symmetric OP.

2. **A differential equation for elliptic polynomials.** By a modification of the method of Heine [8] we obtained a differential equation for elliptic orthogonal polynomials with the weight function $X^{-\frac{1}{2}}$. The coefficients depended upon one parameter only, thus avoiding the unmanageable algebraic equations. We shall apply here the more direct and simpler method of Shohat [15].

By (29), we may write

$$X\phi_n' = nk^2\phi_{n+3} + h_1\phi_{n+1} + h_2\phi_{n-1} + h_3\phi_{n-3}\,, \tag{36}$$

h_i $(i = 1, 2, 3)$ being constants, determined by (28)–(35) so that (36) becomes

$$\begin{aligned} X\phi_n' = nk^2\phi_{n+3} &+ \{nk^2(\lambda_{n+1} + \lambda_{n+2} + \lambda_{n+3}) + 2k^2(\lambda_2 + \lambda_3 + \cdots + \lambda_n) \\ &- n(1 + k^2)\}\phi_{n+1} + \lambda_{n+1}\{n(1 + k^2) - (n - 1)k^2(\lambda_n + \lambda_{n+1} + \lambda_{n+2}) \\ &- 2k^2(\lambda_2 + \lambda_3 + \cdots + \lambda_{n+2})\}\phi_{n-1} - (n - 1)k^2\lambda_{n-1}\lambda_n\lambda_{n+1}\phi_{n-3}\,. \end{aligned}$$

Using the recurrence relation (20), with $c_n = 0$, we may write (36) as

$$X\phi_n' = (L_n x + \lambda_{n+1}M_n)\phi_{n-1} - \lambda_n L_n\phi_{n-2}\,, \tag{37}$$

$$L_n \equiv nk^2x^3 + \{(2n - 1)k^2\lambda_{n+1} - n(1 + k^2) + 2k^2(\lambda_2 + \cdots + \lambda_n)\}x\,, \tag{38}$$

$$M_n \equiv -(2n - 1)k^2x^2 - H_n\,, \qquad M_n' \equiv N_n \equiv -2(2n - 1)k^2x^2\,, \tag{39}$$

$$H_n \equiv (2n + 1)k^2(\lambda_{n+1} + \lambda_{n+2}) - 2n(1 + k^2) + 4k^2(\lambda_2 + \cdots + \lambda_n). \tag{40}$$

By differentiating (37) and using it to eliminate $\phi_{n-1}'(x)$, we get

$$\begin{aligned} &X^2\phi_n'' + X(X' - L_n)\phi_n' - XL_n' \\ &\qquad = \lambda_{n+1}(N_nX + M_nL_{n-1})\phi_{n-1} + \lambda_n\lambda_{n+1}M_nM_{n-1}\phi_{n-2}\,. \end{aligned} \tag{41}$$

Eliminate ϕ_{n-1} and ϕ_{n-2} from (37), (41) and the recurrence relation, and we obtain the desired differential equation

$$\begin{aligned} &M_nX\phi_n'' + \{M_nX' - N_nX - M_n(L_n + L_{n-1}) - M_nM_{n-1}x\}\phi_n' \\ &\quad + \left\{L_nN_n - M_nL_n' + \frac{M_n}{X^{\frac{1}{2}}}(L_nL_{n-1} + M_{n-1}L_nx + \lambda_{n+1}M_nM_{n-1})\right\}\phi_n = 0. \end{aligned} \tag{42}$$

For $k = 0$, (42) reduces to

$$(1 - x^2)\phi_n'' - x\phi_n' + n^2\phi_n = 0,$$

the differential equation for the trigonometric polynomials $T_n(x) = \cos n \arccos x$.

A modification of the method used above [15; 405], together with (28)–(36), gives further

$$X\phi_n' - L_n\phi_n = M_n\phi_{n-1}\,, \tag{43}$$

$$X\phi_n'' + \tfrac{1}{2}X'\phi_n' - C_n\phi_n = A_n\phi_{n-1} + \lambda_n B_n\phi_{n-2}\,, \tag{44}$$

where

$$A_n = n(n+1)k^2\{x^3 - \lambda_{n+1}x + \lambda_{n+2}x\},$$

$$B_n = -n(n+1)k^2(x^2 - \lambda_{n+2}) + (n-1)(n-2)\lambda_{n+1}x^2,$$

$$C_n = (n-1)(n-2)k^2(\lambda_{n+1} + \lambda_{n+2}) - 2(2n-1)k^2 d_{n+2,n} - n^2(1+k^2).$$

Eliminate ϕ_{n-1} and ϕ_{n-2} from the recurrence relation and (43), (44), and we get the differential equation in a new form

$$\begin{aligned} M_nX\phi_n'' + \{\tfrac{1}{2}M_nX' - X(A_n + B_nx)\}\phi_n' \\ +\{L_n(A_n + B_nx) + M_n(B_n - C_n)\}\phi_n = 0. \end{aligned} \tag{45}$$

Observe that

$$A_n + B_nx = -2(2n-1)k^2\lambda_{n+1}x \equiv N_n\,. \tag{46}$$

Let

$$\begin{aligned} D_n \equiv B_n - C_n = -n(n+1)k^2x^2 \\ -2(2n-1)k^2(\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}) + n^2(1+k^2). \end{aligned} \tag{47}$$

The differential equation (45) becomes

$$M_nX\phi_n'' + \{\tfrac{1}{2}M_nX' - N_nX\}\phi_n' + \{L_nN_n + M_nD_n\}\phi_n = 0. \tag{48}$$

We conclude since (42) and (48) must be identical, that is,

$$L_n + L_{n-1} + M_{n-1}x = \tfrac{1}{2}X', \tag{49}$$

$$L_n(\tfrac{1}{2}X' - L_n) + \lambda_{n+1}M_nM_{n-1} \equiv X(L_n' + D_n). \tag{50}$$

It will be observed that the coefficients in the above differential equation for elliptic OP are dependent upon λ_i $(i = 1, 2, \cdots, n)$ only.

3. **Recurrence relation for λ_i.** Write the left member of the second relation of (32), using (28), in the form

$$\int_{-1}^{1} \phi_{n+1}(x)[nk^2x^{n+3} - n(1+k^2)x^{n+1} + (n-2)k^2 d_{n,n-2}x^{n+1}]\frac{dx}{X^{\frac{1}{2}}}, \tag{51}$$

all other terms being zero by the orthogonality relations. By (30), the integral (51) becomes

$$\int_{-1}^{1} \phi_{n+1}(x)\phi_n'(x)X^{\frac{1}{2}}\,dx = \frac{1}{a_{n+1}^2}\{(n-2)k^2 d_{n,n-2} - nk^2 d_{n+3,n+1} - n(1+k^2)\}. \tag{52}$$

Hence, the right members of (52) and of the second relation in (32) are equal, that is,

$$\begin{aligned} 4k^2 d_{n+1,n-3} &= k^2\lambda_{n+2}\{(2n+1)\lambda_{n+1} + (2n+1)\lambda_{n+2} + (2n+3)\lambda_{n+3} + n + 1\} \\ &\quad - 2n(1+k^2)\lambda_{n+2} + 2(1+k^2)d_{n+2,n} + 2k^2(\lambda_{n+3} + \lambda_{n+4})d_{n+2,n} \\ &\quad + 2k^2 d_{n+4,n+2}d_{n+2,n}\,. \end{aligned} \tag{53}$$

Since

$$d_{n+2,n}^2 = d_{n+2,n}d_{n+4,n} + (\lambda_{n+3} + \lambda_{n+4})d_{n+2,n}\,,$$

(53) may be written as

$$\begin{aligned} 4k^2 d_{n+1,n-3} &= \lambda_{n+2}\{(2n+1)k^2(\lambda_{n+1} + \lambda_{n+2}) + (2n+3)k^2\lambda_{n+3} \\ &\quad - 2n(1+k^2)\} + n + 1 + 2(1+k^2)d_{n+2,n} + 2k^2 d_{n+2,n}^2\,. \end{aligned} \tag{54}$$

Change n to $n-1$ in (54) and subtract, using (34) and (35), also

$$d_{n,n-2}^2 - d_{n+1,n-1}^2 = 2\lambda_{n+1}d_{n,n-2} - \lambda_{n+1}^2\,,$$

to get

$$\begin{aligned} &\lambda_n[(2n-7)k^2\lambda_{n-1} + (2n-7)k^2\lambda_n - 2(n-2)(1+k^2) - 4k^2 d_{n,n-2}] \\ &\quad - \lambda_{n+1}[(2n+1)k^2\lambda_{n+1} + (2n+1)k^2\lambda_{n+2} - 2n(1+k^2) - 4k^2 d_{n,n-2}] = 1, \end{aligned}$$

and finally,

$$\lambda_n H_{n-2} = \lambda_{n+1}H_n + 1 \qquad (n \geq 3), \tag{55}$$

where H_n are given by (40).

All λ_n can now be determined successively, using

$$\lambda_1 = 2K, \qquad \lambda_2 = \frac{K - E_1}{k^2 K}, \qquad \lambda_3 = \frac{(3-4k^2)K^2 - 2(1-4k^2)KE_1 - 5E_1^2}{10k^2 K(K - E_1)}.$$

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